4 Film Extension of the Dynamics: Slowness as Stability

4.1 Equation for the Film Motion

One of the difficulties in the problem of reducing the description is caused by the fact that there exists no commonly accepted formal definition of slow (and stable) positively invariant manifolds. Classical definitions of stability and asymptotic stability of the invariant sets sound as follows: Let a dynamical system be defined in some metric space (so that we can measure distances between points), and let $x(t, x_0)$ be a motion of this system at time t with the initial condition $x(0) = x_0$ at time $t = 0$. The subset S of the phase space is called *invariant* if it is made of whole trajectories, that is, if $x_0 \in S$ then $x(t, x_0) \in S$ for all $t \in (-\infty, \infty)$.

Let us denote as $\rho(x, y)$ the distance between the points x and y. The distance from x to a closed set S is defined as usual: $\rho(x, S) = \inf\{\rho(x, y)|y \in S\}$. The closed invariant subset S is called *stable*, if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\rho(x_0, S) < \delta$, then for every $t > 0$ it holds $\rho(x(t, x_0), S) < \epsilon$. A closed invariant subset S is called *asymptotically stable* if it is stable and attractive, that is, there exists $\epsilon > 0$ such that if $\rho(x_0, S) < \epsilon$, then $\rho(x(t, x_0), S) \to 0$ as $t \to \infty$.

Formally, one can reiterate the definitions of stability and of the asymptotic stability for positively invariant subsets. Moreover, since in the definitions mentioned above it goes only about $t \geq 0$ or $t \to \infty$, it might seem that positively invariant subsets can be a natural object of study concerning stability issues. Such conclusion is misleading, however. The study of the classical stability of the positively invariant subsets reduces essentially to the notion of stability of invariant sets – maximal attractors.

Let Y be a closed positively invariant subset of the phase space. *The maximal attractor* for Y is the set M_Y,

$$M_Y = \bigcap_{t \geq 0} T_t(Y) , \tag{4.1}$$

where T_t is the shift operator for the time t:

$$T_t(x_0) = x(t, x_0) .$$

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The maximal attractor M_Y is invariant, and the stability of Y defined classically is equivalent to the stability of M_Y under any sensible assumption about uniform continuity (for example, it is so for a compact phase space).

For systems which relax to a stable equilibrium, the maximal attractor is simply one and the same for any bounded positively invariant subset, and it consists of a single stable point.

It is important to note that in the definition (4.1) one considers motions of a positively invariant subset to equilibrium *along itself*: $T_t Y \subset Y$ for $t \geq 0$. It is precisely this motion which is uninteresting from the perspective of the comparison of stability of positively invariant subsets. If one subtracts this *motion along itself* out of the vector field $J(x)$ (3.1), one obtains a less trivial picture.

We again assume submanifolds in U parameterized with a single parameter set $F: W \to U$. Note that there exists a wide class of transformations which do not alter the geometric picture of motion: For a smooth diffeomorphism $\varphi : W \to W$ (a smooth coordinate transform), maps F and $F \circ \varphi$ define the same geometric pattern in the phase space.

Let us consider motions of the manifold $F(W)$ along solutions of equation (3.1). Denote as F_t the time-dependent map, and write equation of motion for this map:

$$\frac{\mathrm{d}F_t(y)}{\mathrm{d}t} = J(F_t(y)) . \tag{4.2}$$

Let us now subtract the component of the vector field responsible for the motion of the map $F_t(y)$ along itself from the right hand side of equation (4.2). In order to do this, we decompose the vector field $J(x)$ in each point $x = F_t(y)$ as

$$J(x) = J_{\parallel}(x) + J_{\perp}(x) , \tag{4.3}$$

where $J_{\parallel}(x) \in T_{t,y}$ ($T_{t,y} = (D_y F_t(y)(L))$. If projectors are well defined, $P_{t,y} = P(F_t(y), T_{t,y})$, then decomposition (4.3) has the form:

$$J(x) = P_{t,y} J(x) + (1 - P_{t,y}) J(x) . \tag{4.4}$$

Subtracting the component $J_{\parallel}$ from the right hand side of equation (4.2), we obtain,

$$\frac{\mathrm{d}F_t(y)}{\mathrm{d}t} = (1 - P_{t,y}) J(F_t(y)) . \tag{4.5}$$

Note that the geometric pictures of motion corresponding to equations (4.2) and (4.5) are identical *locally* in y and t. Indeed, the infinitesimal shift of the manifold W along the vector field is easily computed:

$$(D_y F_t(y))^{-1} J_{\parallel}(F_t(y)) = (D_y F_t(y))^{-1} (P_{t,y} J(F_t(y))) . \tag{4.6}$$

This defines a smooth change of the coordinate system (assuming all solutions exist). In other words, the component $J_{\perp}$ defines the motion of the manifold in U, while we can consider (locally) the component $J_\parallel$ as a component which locally defines motions in W (a coordinate transform).

The positive semi-trajectory of motion (for $t > 0$) of any submanifold in the phase space along the solutions of initial differential equation (3.1) (without subtraction of $J_\parallel(x)$) is the positively invariant manifold. The closure of such semi-trajectory is an invariant subset. The construction of the invariant manifold as a trajectory of an appropriate initial edge may be useful for producing invariant exponentially attracting set [173, 174]. Very recently, the notion of exponential stability of invariants manifold for ODEs was revised by splitting motions into tangent and transversal (orthogonal) components in [175].

We further refer to equation (4.5) as *the film extension* of the dynamical system (3.1). The phase space of the dynamical system (4.5) is the set of maps F (films). Fixed points of equation (4.5) are solutions to the invariance equation in the differential form (3.3). These include, in particular, all positively invariant manifolds. Stable or asymptotically stable fixed points of equation (4.5) are the slow manifolds we are interested in. It is the notion of stability associated with the film extension of the dynamics which is relevant to our study. In Chap. 9, we consider relaxation methods for constructing slow positively invariant manifolds on the basis of the film extension (4.5).

4.2 Stability of Analytical Solutions

When studying the Cauchy problem for equation (4.5), one should ask a question of how to choose the boundary conditions the function F must satisfy at the boundary of W. Without fixing the boundary conditions, the general solution of the Cauchy problem for the film extension equations (4.5) in the class of smooth functions on W is essentially ambiguous.

The boundary of W, ∂W, splits in two pieces: $\partial W = \partial W_+ \bigcup \partial W_-$. For a smooth boundary these parts can be defined as

$$\begin{aligned} \partial W_+ &= \{y \in \partial W | (\nu(y), (DF(y))^{-1}(P_y J(F(y)))) < 0\} \, , \\ \partial W_- &= \{y \in \partial W | (\nu(y), (DF(y))^{-1}(P_y J(F(y)))) \geq 0\} \, . \end{aligned} \tag{4.7}$$

where $\nu(y)$ denotes the unit outer normal vector at the boundary point y, and $(DF(y))^{-1}$ is the isomorphism of the tangent space T_y on the linear space of parameters L.

One can understand the boundary splitting (4.7) in such a way: The projected vector field $P_y J(F(y))$ defines dynamics on the manifold $F(W)$, this dynamics is the image of some dynamics on W. The corresponding vector field on W is $v(y) = (DF(y))^{-1}(P_y J(F(y)))$. The boundary part ∂W_+ consists of points y, where the velocity vector $v(y)$ points inside W, while for $y \in \partial W_-$ this vector $v(y)$ is directed outside of W (or is tangent to ∂W). The splitting $\partial W = \partial W_+ \bigcup \partial W_-$ depends on t with the vector field $v(y)$:

$$v_t(y) = (DF_t(y))^{-1}(P_y J(F_t(y))) ,$$

and the dynamics of $F_t(y)$ is determined by (4.5).

If we would like to derive a solution of the film extension (4.5) $F(y,t)$ for $(y,t) \in W \times [0,\tau]$ for some time $\tau > 0$, then it is necessary to fix some boundary conditions on ∂W_+ (for the "incoming from abroad" part of the function $F(y)$).

Nevertheless, there is a way to study equation (4.5) in W without introducing any boundary condition. It is in the spirit of the classical Cauchy-Kovalevskaya theorem [176–178] about analytical Cauchy problem solutions with analytical data, as well as in the spirit of the classical Lyapunov auxiliary theorem about analytical invariant manifolds in the neighborhood of a fixed point [3,52] and the Poincaré theorem [50] about analytical linearization of analytical non-resonant contractions (see [181]).

We note in passing that recently the interest to the classical analytical Cauchy problem is revived in the mathematical physics literature [179,180]. In particular, analogs of the Cauchy-Kovalevskaya theorem were obtained for the generalized Euler equations [179]. A technique to estimate the convergence radii of the series emerging therein was also developed.

Analytical solutions to equation (4.5) do not require boundary conditions on the boundary of W. The analyticity condition itself allows finding unique analytical solutions of the equation (4.5) with the analytical right hand side $(1-P)J$ for analytical initial conditions F_0 in W (assuming that such solutions exist). Of course, the analytical continuation without additional regularity conditions is an ill-posed problem. However, it may be useful to switch from functions to germs[1]: we can solve chains of ordinary differential equations for Taylor coefficients instead of partial differential equations for functions (4.5), and it may be possible to prove the convergence of the Taylor series thus obtained. This is the way to prove the Lyapunov auxiliary theorem [3], and one of the known ways to prove the Cauchy-Kovalevskaya theorem.

Let us consider the system (3.1) with stable equilibrium point x^*, real analytical right hand side J, and real analytical projector field $P(x,T)\colon E \to T$. We shall study real analytical sub-manifolds, which include the equilibrium point point x^* $(0 \in W, F(0) = x^*)$. Let us expand F in a Taylor series in the neighborhood of zero:

$$F(y) = x^* + A_1(y) + A_2(y,y) + \ldots + A_k(y,y,\ldots,y) + \ldots \quad , \tag{4.8}$$

where $A_k(y,y,\ldots,y)$ is a symmetric k-linear operator $(k = 1,2,\ldots)$.

Let us expand also the right hand side of the film equation (4.5). Matching operators of the same order, we obtain a hierarchy of equations for $A_1,\ldots,A_k,\ldots$:

[1] The germ is the sequences of Taylor coefficients that represent an analytical function near a given point.

$$\frac{\mathrm{d}A_k}{\mathrm{d}t} = \Psi_k(A_1, \ldots, A_k) \ . \tag{4.9}$$

It is crucially important, that the dynamics of A_k does not depend on $A_{k+1}, \ldots$, and equations (4.9) can be studied in the following order: we first study the dynamics of A_1, then the dynamics of A_2 with the A_1 motion already given, then A_3 and so on.

Let the projector P_y in equation (4.5) be an analytical function of the derivative $D_y F(y)$ and of the deviation $x - x^*$. Let the corresponding Taylor series expansion at the point $(A_1^0(\bullet), x^*)$ have the form:

$$D_y F(y)(\bullet) = A_1(\bullet) + \sum_{k=2}^{\infty} k A_k(y, \ldots, \bullet) \ , \tag{4.10}$$

$$P_y = \sum_{k,m=0}^{\infty} P_{k,m}(\underbrace{D_y F(y)(\bullet) - A_1^0(\bullet), \ldots, D_y F(y)(\bullet) - A_1^0(\bullet)}_{k} ;$$
$$\underbrace{F(y) - x^*, \ldots, F(y) - x^*}_{m}) \ ,$$

where $A_1^0(\bullet)$, $A_1(\bullet)$, $A_k(y, \ldots, \bullet)$ are linear operators. $P_{k,m}$ is a $k+m$-linear operator $(k, m = 0, 1, 2, \ldots)$ with values in the space of linear operators $E \to E$. The operators $P_{k,m}$ depend on the operator $A_1^0(\bullet)$ as on a parameter. Let the point of expansion $A_1^0(\bullet)$ be the linear part of F: $A_1^0(\bullet) = A_1(\bullet)$.

Let us represent the analytical vector field $J(x)$ as a power series:

$$J(x) = \sum_{k=1}^{\infty} J_k(x - x^*, \ldots, x - x^*) \ , \tag{4.11}$$

where J_k is a symmetric k-linear operator $(k = 1, 2, \ldots)$.

Let us write, for example, the first two equations of the equation chain (4.9):

$$\frac{\mathrm{d}A_1(y)}{\mathrm{d}t} = (1 - P_{0,0})J_1(A_1(y)) \ ,$$
$$\frac{\mathrm{d}A_2(y,y)}{\mathrm{d}t} = (1 - P_{0,0})[J_1(A_2(y,y)) + J_2(A_1(y), A_1(y))]$$
$$-[2P_{1,0}(A_2(y,\bullet)) + P_{0,1}(A_1(y))]J_1(A_1(y)) \ . \tag{4.12}$$

Here, operators $P_{0,0}$, $P_{1,0}(A_2(y,\bullet))$, $P_{0,1}(A_1(y))$ parametrically depend on the operator $A_1(\bullet)$; hence, the first equation is nonlinear, and the second is linear with respect to $A_2(y,y)$. The leading term in the right hand side has the same form for all equations of the sequence (4.9):

$$\frac{\mathrm{d}A_n(y, \ldots, y)}{\mathrm{d}t} \tag{4.13}$$
$$= (1 - P_{0,0})J_1(A_n(y, \ldots, y)) - nP_{1,0}(A_n(\underbrace{y, \ldots, y}_{n-1}, \bullet))J_1(A_1(y)) + \ldots \ .$$

There are two important conditions on P_y and $D_yF(y)$: $P_y^2 = P_y$, because P_y is a projector, and $\mathrm{im}P_y = \mathrm{im}D_yF(y)$, because P_y projects on the image of $D_yF(y)$. If we expand these conditions in the power series, then we get the conditions on the coefficients. For example, from the first condition we get:

$$
\begin{aligned}
&P_{0,0}^2 = P_{0,0}\ ,\\
&P_{0,0}[2P_{1,0}(A_2(y,\bullet)) + P_{0,1}(A_1(y))] + [2P_{1,0}(A_2(y,\bullet)) + P_{0,1}(A_1(y))]P_{0,0}\\
&= 2P_{1,0}(A_2(y,\bullet)) + P_{0,1}(A_1(y)), \dots\ .
\end{aligned}
\tag{4.14}
$$

After multiplication of the second equation in (4.14) with $P_{0,0}$ we get

$$
P_{0,0}[2P_{1,0}(A_2(y,\bullet)) + P_{0,1}(A_1(y))]P_{0,0} = 0\ . \tag{4.15}
$$

Similar identities can be obtained for any oder of the expansion. These equalities allow us to simplify the stationary equation for the sequence (4.9). For example, for the first two equations of the sequence (4.12) we obtain the following stationary equations:

$$
\begin{aligned}
&(1 - P_{0,0})J_1(A_1(y)) = 0\ ,\\
&(1 - P_{0,0})[J_1(A_2(y,y)) + J_2(A_1(y), A_1(y))]\\
&-[2P_{1,0}(A_2(y,\bullet)) + P_{0,1}(A_1(y))]J_1(A_1(y)) = 0\ .
\end{aligned}
\tag{4.16}
$$

The operator $P_{0,0}$ is the projector on the space $\mathrm{im}A_1$ (the image of A_1), hence, from the first equation in (4.16) it follows: $J_1(\mathrm{im}A_1) \subseteq \mathrm{im}A_1$. So, $\mathrm{im}A_1$ is a J_1-invariant subspace in E ($J_1 = D_xJ(x)|_{x^*}$) and $P_{0,0}(J_1(A_1(y)) \equiv J_1(A_1(y)$. It is equivalent to the first equation of (4.16). Let us multiply the second equation of (4.16) with $P_{0,0}$ from the left. As a result we obtain the condition:

$$
P_{0,0}[2P_{1,0}(A_2(y,\bullet)) + P_{0,1}(A_1(y))]J_1(A_1(y)) = 0\ ,
$$

for solution of equations (4.16), because $P_{0,0}(1 - P_{0,0}) \equiv 0$. If $A_1(y)$ is a solution of the first equation of (4.16), then this condition becomes an identity, and we can write the second equation of (4.16) in the form

$$
\begin{aligned}
&(1 - P_{0,0})[J_1(A_2(y,y)) + J_2(A_1(y), A_1(y)) - (2P_{1,0}(A_2(y,\bullet))\\
&+P_{0,1}(A_1(y)))J_1(A_1(y))] = 0\ .
\end{aligned}
\tag{4.17}
$$

It should be stressed, that the choice of the projector field P_y (4.10) has impact only on the $F(y)$ parametrization, whereas the invariant geometrical properties of the solutions of (4.5) do not depend on the projector field if some transversality and analyticity conditions hold. The conditions of thermodynamic structures preservation significantly reduce ambiguity of the projector choice. One of the most important condition is $\ker P_y \subset \ker D_xS$, where $x = F(y)$ and S is the entropy (see Chap. 5 about the entropy below). The thermodynamic projector is the unique operator which transforms the arbitrary vector field equipped with the given Lyapunov function into a vector field with the same Lyapunov function on the arbitrary submanifold which is not tangent to the level of the Lyapunov function. For the thermodynamic projectors P_y the entropy $S(F(y))$ is conserved on the solutions $F(y,t)$ of the equation (4.5) for any $y \in W$.

If the projectors P_y in equations (4.10)–(4.17) are thermodynamic, then $P_{0,0}$ is the orthogonal projector with respect to the entropic scalar product[2]. For orthogonal projectors the operator $P_{1,0}$ has a simple explicit form. Let $A : L \to E$ be an isomorphic injection (an isomorphism on the image), and $P : E \to E$ be the orthogonal projector on the image of A. The orthogonal projector on the image of the perturbed operator $A + \delta A$ is $P + \delta P$,

$$\begin{aligned} \delta P &= (1-P)\delta A A^{-1} P + (\delta A A^{-1} P)^{+}(1-P) + o(\delta A), \\ P_{1,0}(\delta A(\bullet)) &= (1-P)\delta A(\bullet) A^{-1} P + (\delta A(\bullet) A^{-1} P)^{+}(1-P)\,. \end{aligned} \tag{4.18}$$

In (4.18), the operator A^{-1} is defined on $\mathrm{im}A$, $\mathrm{im}A = \mathrm{im}P$, and the operator $A^{-1}P$ acts on E.

Equation (4.18) for δP follows from the three conditions:

$$(P+\delta P)(A+\delta A) = A+\delta A,\, (P+\delta P)^2 = P+\delta P,\, (P+\delta P)^{+} = P+\delta P\,. \tag{4.19}$$

Every A_k is driven by $A_1, \dots, A_{k-1}$. Stability of the germ of the positively invariant analytical manifold $F(W)$ at point 0 ($F(0) = x^*$) is defined as stability of the solution of the corresponding equations sequence (4.9). Moreover, the notion of the k-jet stability can be useful: let us call k-jet stable such a germ of a positively invariant manifold $F(M)$ at the point 0 ($F(0) = x^*$), if the corresponding solution of the equation sequence (4.9) is stable for $k = 1, \dots, n$. The simple “triangle” structure of the equation sequence (4.9) with the form (4.13) of principal linear part makes the problem of jets stability very similar for all orders $n > 1$.

Let us demonstrate the stability conditions for the 1-jets in a n-dimensional space E. Let the Jacobian matrix $J_1 = D_x J(x)|_{x^*}$ be selfadjoint with a simple spectrum $\lambda_1, \dots, \lambda_n$, and the projector $P_{0,0}$ be orthogonal (this is a typical “thermodynamic” situation). The eigenvectors of J_1 form a basis in E: $\{e_i\}_{i=1}^n$. Let a linear space of parameters L be the k-dimensional real space, $k < n$. We shall study the stability of operator A_1^0 which is a fixed point for the first equation of the sequence (4.9). The operator A_1^0 is a fixed point of this equation, if $\mathrm{im}A_1^0$ is a J_1-invariant subspace in E. We discuss full-rank operators, so, for some order of $\{e_i\}_{i=1}^n$ numbering, the matrix of A_1^0 should have a form: $a_{1ij}^0 = 0$, if $i > k$. Let us choose the basis in L: $l_j = (A_1^0)^{-1} e_j$, ($j = 1, \dots, k$). For this basis $a_{1ij}^0 = \delta_{ij}$, ($i = 1, \dots, n$, $j =$

[2] This scalar product is the bilinear form defined by the negative second differential of the entropy at the point x^*, $-D^2S(x)$.

$1, \ldots, k$, where δ_{ij} is the Kronecker symbol). The corresponding projectors P and $1-P$ have the matrices:

$$P = \mathrm{diag}(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}), \; 1 - P = \mathrm{diag}(\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{n-k}) \; , \tag{4.20}$$

where $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ is the $n \times n$ diagonal matrix with numbers $\alpha_1, \ldots, \alpha_n$ on the diagonal.

The equations of the linear approximation for the dynamics of the variation δA read:

$$\frac{\mathrm{d}\delta A}{\mathrm{d}t} = \mathrm{diag}(\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{n-k})[\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\delta A - \delta A \mathrm{diag}(\underbrace{\lambda_1, \ldots, \lambda_k}_{k})] \; . \tag{4.21}$$

The time derivative of A is orthogonal to A: for any $y, z \in L$ the equality $(\dot{A}(y), A(x)) = 0$ holds, hence, for the stability analysis it is necessary and sufficient to study δA with $\mathrm{im}\delta A_1^0 \perp \mathrm{im}A$. The matrix for such a δA has the form:

$$\delta a_{ij} = 0, \; \mathrm{if} \; i \leq k \; .$$

For $i = k+1, \ldots, n$, $j = 1, \ldots, k$ equation (4.21) gives:

$$\frac{\mathrm{d}\delta a_{ij}}{\mathrm{d}t} = (\lambda_i - \lambda_j)\delta a_{ij} \; . \tag{4.22}$$

Therefore, the stability condition becomes:

$$\lambda_i - \lambda_j < 0 \; \text{for all} \; i > k, \; j \leq k \; . \tag{4.23}$$

This means that the relaxation *towards* $\mathrm{im}A$ (with the spectrum of relaxation times $|\lambda_i|^{-1}$ $(i = k+1, \ldots, n)$) is faster, than the relaxation *along* $\mathrm{im}A$ (with the spectrum of relaxation times $|\lambda_j|^{-1}$ $(j = 1, \ldots, k)$).

Let the condition (4.23) hold. For negative λ, it means that the relaxation time for the film (in the first approximation) is:

$$\tau = 1/(\min_{i>k} |\lambda_i| - \max_{j \leq k} |\lambda_j|) \; ,$$

thus it depends on the *spectral gap* in the spectrum of the operator $J_1 = D_x J(x)|_{x^*}$.

It is the gap between spectra of two restrictions of the operator J_1, $J_1^{\parallel}$ and $J_1^{\perp}$, respectively. The operator $J_1^{\parallel}$ is the restriction of J_1 on the J_1-invariant subspace $\mathrm{im}A_1^0$ (it is the tangent space to the slow invariant manifold at point x^*). The operator $J_1^{\perp}$ is the restriction of J_1 on the orthogonal complement to $\mathrm{im}A_1^0$. This subspace is also J_1-invariant, because J_1 is selfadjoint. The spectral gap between spectra of these two operators is the spectral gap between relaxation *towards* the slow manifold and relaxation *along* this manifold.

The stability condition (4.23) demonstrates that our formalization of the slowness of manifolds as the stability of fixed points for the film extension (4.5) of initial dynamics meets the intuitive expectations.

For the analysis of system (4.9) in the neighborhood of some manifold F_0 ($F_0(0) = x^*$), the following parametrization can be convenient. Let us consider

$$F_0(y) = A_1(y) + \dots , \; T_0 = A_1(L)$$

to be a tangent space to $F_0(W)$ at point x^*, $E = T_0 \oplus H$ is the direct sum decomposition.

We shall consider analytical sub-manifolds in the form

$$x = x^* + (y, \Phi(y)) \; , \tag{4.24}$$

where $y \in W_0 \subset T_0$, W_0 is a neighborhood of zero in T_0, $\Phi(y)$ is an analytical map of W_0 in H, and $\Phi(0) = 0$. Any analytical manifold close to F_0 can be represented in this form.

Let us define the projector P_y that corresponds to the decomposition (4.24), as the projector on T_y parallel to H. Furthermore, let us introduce the corresponding decomposition of the vector field $J = J_y \oplus J_z$, $J_y \in T_0$, $J_z \in H$. Then

$$P_y(J) = (J_y, (D_y\Phi(y))J_y) \; . \tag{4.25}$$

The corresponding equation of motion of the film (4.5) has the following form:

$$\frac{\mathrm{d}\Phi(y)}{\mathrm{d}t} = J_z(y, \Phi(y)) - (D_y\Phi(y))J_y(y, \Phi(y)) \; . \tag{4.26}$$

If J_y and J_z depend analytically on their arguments, then from (4.26) one can easily obtain a hierarchy of equations of the form (4.9) (of course, $J_y(x^*) = 0, \; J_z(x^*) = 0$).

Using these notions, it is convenient to formulate the *Lyapunov auxiliary theorem* [3]. Let $T_0 = R^m, H = R^p$, and in U an analytical vector field be defined $J(y,z) = J_y(y,z) \oplus J_z(y,z)$, $(y \in T_0, z \in H)$. Assume the following conditions are satisfied:

1. $J(0,0) = 0$;
2. $D_z J_y(y,z)\big|_{(0,0)} = 0$;
3. $0 \notin \mathrm{conv}\{k_1, \dots, k_m\}$,
   where $k_1, \dots, k_m$ are the eigenvalues of the operator $D_y J_y(y,z)\big|_{(0.0)}$, and $\mathrm{conv}\{k_1, \dots, k_m\}$ is the convex hull of $\{k_1, \dots, k_m\}$;
4. the numbers k_i and λ_j are not related by any equation of the form
   $$\sum_{i=1}^{m} m_i k_i = \lambda_j \; , \tag{4.27}$$
   where λ_j $(j = 1, \dots, p)$ are eigenvalues of $D_z J_z(y,z)\big|_{(0,0)}$, and $m_i \geq 0$ are integers, $\sum_{i=1}^{m} m_i > 0$.

Let us also consider an analytical manifold $(y, \Phi(y))$ in U in the neighborhood of zero ($\Phi(0) = 0$) and write for it the differential invariance equation with the projector (4.25):

$$(D_y\Phi(y))J_y(y, \Phi(y)) = J_z(y, \Phi(y)) \ . \tag{4.28}$$

Lyapunov auxiliary theorem. Given conditions 1-4, equation (4.24) has the unique analytical solution in the neighborhood of zero, satisfying the condition $\Phi(0) = 0$.

Recently, various new applications of this theorem were developed [52, 184–186].

In order to weaken the non-resonance condition in [49] the existence of invariant manifolds near fixed points tangent to invariant subspaces of the linearization was proved without assumption that the corresponding space for the linear map is a spectral subspace. (This proof was based on the graph transform method [46].)

Studying germs of invariant manifolds using Taylor series expansion in a neighborhood of a fixed point is definitely useful from the theoretical as well as from the practical perspective. But the well known difficulties pertinent to this approach, of convergence, of small denominators (connected with proximity to the resonances (4.27)) and others call for development of different methods. A hint can be found in the famous KAM theory: one should use iterative methods instead of the Taylor series expansion [4–6]. Below we present two such methods:

– The Newton method subject to incomplete linearization;
– The relaxation method which is the Galerkin-type approximation to Newton's method with projection on the defect of invariance (3.3), i.e. on the right hand side of equation (4.5).